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# ON THE DURATION OF CONTACT FOR THE HERTZIAN IMPACT OF A SPHERICAL INDENTER ON A MAXWELL SOLID

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**Abstract**—An expression is derived for the duration of contact for a Hertzian impact of a spherical indenter on a Maxwell solid valid for $\eta T_0 \ll 1$. The result disagrees with a previous analysis. The expression is compared with numerical predictions of the contact duration in which empirical creep and relaxation functions were used.

## INTRODUCTION

The present note is concerned with the prediction of the duration of contact from the governing equations of motion for the nearly elastic Hertzian impact of a spherical indenter on a Maxwell solid. Hunter[1] produced the first general solution of the impact problem for a linear viscoelastic solid with arbitrary mechanical properties for the case where the penetration $\alpha(t)$ attained a single maximum. The result was a pair of coupled nonlinear integro-differential equations whose solution in practice was feasible only on a numerical basis. However, Hunter analyzed his equations assuming a nearly elastic impact on a Maxwell solid. His resulting equations of motion for the indenter in dimensionless variables valid for $K \ll 1$ were for $\zeta < \zeta_m$

$$\ddot{w} + K(\dot{w} - 1) = -5/4w^{3/2}. \qquad w = 0, \quad \dot{w} = 1, \quad \zeta = 0 \tag{1}$$

and for $\zeta > \zeta_m$

$$\ddot{w} - K(3\dot{w} + 1) = -5/4w^{3/2}. \qquad w = w_m, \quad \dot{w} = 0, \quad \zeta = \zeta_m \tag{2}$$

where $K = \eta\alpha_0/V$, $\dot{w} = \mathrm{d}w/\mathrm{d}\zeta$, $w = r_1^2/R\alpha_0$ is the normalized radius of contact, $\zeta = Vt/\alpha_0$ is the normalized time, and $\zeta_m$ is the normalized time at maximum penetration. Here, $\eta$ is the mean inverse relaxation time of the solid, $R$ is the sphere radius, $r_1$ is the radius of contact, $V$ is the initial impact velocity, and $\alpha_0$ is the maximum penetration for an elastic solid.

Hunter approximated the solutions to equations (1) and (2) giving estimates of the coefficient of restitution and impact duration $T$ valid for $\eta T_0 \ll 1$ where $T_0$ is the elastic impact duration. His results were

$$T = T_0(1 - 0{\cdot}037(\eta T_0)) \tag{3}$$

and

$$e = 1 - 4/9(\eta T_0) \tag{4}$$

where

$$T_0 = 2{\cdot}94\left(\frac{15M(1-\nu)}{32R^{1/2}\mu_D}\right)^{2/5} V^{-1/5} = 2I_2\frac{\alpha_0}{V}.$$

Here, $\mu_D$ is the instantaneous shear modulus, $M$ is the sphere mass, $\nu$ is Poisson's ratio, and $I_2$ is the Hertz integral, $I_2 = \int_0^1 \mathrm{d}y/(1-y^{5/2})^{1/2} \simeq 1{\cdot}47$.

Calvit[2] solved Hunter's original pair of coupled nonlinear integro-differential equations numerically using empirical values of the creep and relaxation functions at various temperatures for polymer blocks to predict experiments for the rebound of steel balls on the same material. His numerical results predicted impact durations $T$ and coefficients of restitution $e$. At all temperatures his solutions gave penetration times $T > T_0$ which is inconsistent with equation (3).

## ANALYSIS

To correct the inconsistency, one can re-examine the analysis of equations (1) and (2) leading to equation (3). Thus, assuming that $K \ll 1$ and $w_m = 1 + \Omega$ where $\Omega = 4/5K(1 - I_1)$ (see [1]), $I_1 = \int_0^1 (1-y^{5/2})^{1/2}\,\mathrm{d}y \simeq 0{\cdot}817$, and $w_m$ is the normalized radius of contact at maximum penetration, the time to maximum penetration is given by

$$\frac{VTm}{\alpha_0} = \int_0^{w_m}\frac{\mathrm{d}w}{\dot{w}} = \int_0^1\frac{\mathrm{d}w}{\dot{w}} + \int_1^{1+\Omega}\frac{\mathrm{d}w}{\dot{w}} \tag{5}$$

where

$$\dot{w}^2 = {w_m}^{5/2} - w^{5/2} + 2K(w - w_m) + 2K\int_w^{w_m}({w_m}^{5/2} - w^{5/2} + 2K(w - w_m))^{1/2}\,\mathrm{d}w. \tag{6}$$

Expanding the integrand of equation (5) to zero order in $K$ gives

$$\frac{1}{\dot{w}} \simeq \frac{1}{(1-w^{5/2})^{1/2}} + 0(K). \tag{7}$$

Evaluation of the first integral on the right of equation (5) then gives

$$\int_0^1\frac{\mathrm{d}w}{\dot{w}} \simeq \int_0^1\frac{\mathrm{d}w}{(1-w^{5/2})^{1/2}} + 0(K) = I_2 + 0(K). \tag{8}$$

However, with the second integral in equation (5) which Hunter seems to have neglected, one can expand $\dot{w}^2$ around $w_m$ giving to first order in $K$

$$\dot{w}^2 \simeq 5/2(w_m - w) + 0(K^2). \tag{9}$$

Substitution of equation (9) for the second integral on the right of equation (5) then gives

$$\int_1^{1+\Omega}\frac{\mathrm{d}w}{\dot{w}} \simeq \left(\frac{2}{5}\right)^{1/2}\int_1^{1+\Omega}\frac{\mathrm{d}w}{(w_m - w)^{1/2}} + 0(K^{3/2})$$

or

$$\int_1^{1+\Omega}\frac{\mathrm{d}w}{\dot{w}} \simeq 2(2/5)^{1/2}\Omega^{1/2} + 0(K^{3/2}). \tag{10}$$

After carrying the same procedure out above for times $t > T_m$, one finds the impact duration to be

$$\frac{VT}{\alpha_0} = 2I_2 + \frac{8}{5}\sqrt{2}(1 - I_1)^{1/2}K^{1/2} + 0(K) \tag{11}$$

or that

$$T = T_0(1 + 0{\cdot}193(\eta T_0)^{1/2} + 0(\eta T_0)). \tag{12}$$

Equation (12) is now consistent with Calvit's result.

## DISCUSSION

In lieu of solving equations (1) and (2) numerically, the validity of equation (12) was checked using Calvit's numerical results. This was accomplished by making the *a priori* assumption that Calvit's numerical solutions described a nearly elastic collision on a Maxwell solid. Thus, values of $\eta T_0$ for an equivalent Maxwell solid were determined at various temperatures using Calvit's predictions of the coefficient of restitution $e$ and equation (4). A crossplot of the equivalent $\eta T_0$ vs temperature and Calvit's prediction of $T/T_0$ vs temperature was then compared with equation (12). The result is shown in Fig. 1. The agreement is fair particularly if one notes that Calvit acknowledges spurious results in his determination of the relaxation and creep function at 80°C. (This corresponds to $\eta T_0 \sim {\cdot}25$ in Fig. 1.)

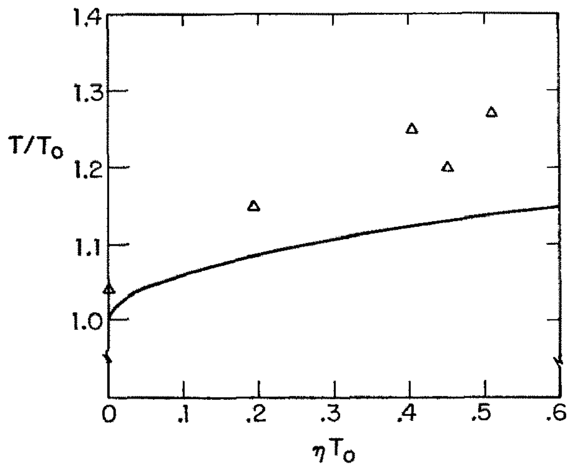


Fig. 1. Impact duration versus mean inverse relaxation time. ———, Eq. (12). Δ, from calculations for 11/16 in. steel sphere dropping onto polymethylmethcrylate[2].

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## REFERENCES

1. S. C. Hunter, *J. Mech. Phys. Solids* **8**, 219 (1960).
2. H. H. Calvit, *Int. J. Solids Struc.* **3**, 951 (1967).

**Резюме** — Получили выражение на продолжительность контакта сферического индентора при герционовском ударе по твердому телу, подчиняющемуся закону Максвелла, в данном случае — $nT_0 \ll 1$. Результат не совпал с прежним анализом. Выражение сравнивается с численными предсказаниями продолжительности контакта в которых использовались эмпирическая деформация ползучести и функция релаксаций.